Risk-Neutral Pricing
Part III - Pricing A Credit Default Swap

Gary Schurman, MBE, CFA

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In Part I of this series we determined that given our matrix of asset prices at time zero and asset payoffs at time one there was an arbitrage available to us such that for an investment of \$0 at time zero there was a certain payoff of \$100 at time one regardless of the state-of-the-world at that time. In Part II of this series we determined the correct no-arbitrage asset prices at time zero and introduced risk-neutral pricing. In Part III of this series we will use the concepts developed in Parts I and II to price a credit default swap. To demonstrate the mathematics we will work through the following hypothetical problem...

Our Hypothetical Problem

The economy in which we operate has three assets. These assets are a risk-free bond, company assets (unlevered) and company debt. The three possible states-of-the-world at time one are ω_a, ω_2 and ω_c. The table below presents the prices of our assets at time zero and the asset payoffs at time one given the state-of-the-world at that time...

Table 1 - Asset Prices and Payoffs

Asset Symbol	Asset Description	Price t = 0	Payoff t = 1 ω_a	ω_b	ω_c
B	Risk-free bond	100	105	105	105
A	Company assets	112	40	120	150
D	Company debt	81	40	90	90

From the table above we can see that an investment in Asset D (Company debt) is not a risk-free investment. In states ω_b and ω_c the value of company assets exceeds the value of company debt such that the investor who holds a long position in Asset D makes a positive return on his or her investment. In state ω_a the value of company assets is only \$40 such that the payoff on Asset D is capped at \$40 and the investor who holds a long position in Asset D makes a negative return on his or her investment. Accordingly this investor will recognize a \$9 gain (\$90 - \$81) in states ω_b and ω_c and a \$41 loss (\$40 - \$81) in state ω_a. If we were to write (i.e. short) a credit default swap (CDS) on Asset D then an investor who is long Asset D and long the CDS would hold a risk-free portfolio. For the portfolio to be risk-free the payoff on the CDS in states ω_a, ω_b and ω_c would have to be \$50, \$0 and \$0, respectively. The table below presents the no-arbitrage price of the CDS at time zero (to be determined) and the CDS payoffs at time one given the state-of-the-world at that time...

Table 2 - CDS Price and Payoffs

Asset Symbol	Asset Description	Price t = 0	Payoff t = 1 ω_a	ω_b	ω_c
S	CDS	TBD	50	0	0

Question: Given the time zero asset prices and payoffs in Table 1 and the CDS payoffs in Table 2, what is the time zero no-arbitrage price of a credit default swap written on the company's debt?

Pricing The Credit Default Swap

The table below presents the time zero **risk-neutral** probabilities of finding ourselves in either state ω_a, state ω_b or state ω_c at time one...

Table 3 - Risk-Neutral Probabilities (Measure Q)

Description	Symbol	Probability
Risk-neutral probability that we will find ourselves in state ω_a at time one	q_a	To be determined
Risk-neutral probability that we will find ourselves in state ω_b at time one	q_b	To be determined
Risk-neutral probability that we will find ourselves in state ω_c at time one	q_c	To be determined

We will define S_a to be the CDS payoff at time one given state ω_a, S_b to be the CDS payoff at time one given state ω_b, S_c to be the CDS payoff at time one given state ω_c and K_b to be the risk-free rate at time zero. Given these definitions and Tables 1 and 2 above the equation for the no-arbitrage (i.e. risk-neutral) price of the CDS at time zero (see Part II) is...

$$S_0 = \mathbb{E}^Q\left[S_T \times \left(1 + K_b\right)^{-1}\right] = \frac{B_0}{B_T}\left(S_a\,q_a + S_b\,q_b + S_c\,q_c\right) \tag{1}$$

The only unknowns in Equation (1) are the risk-neutral probabilities q_a, q_b and q_c. If we can come up with three linearly independent equations that are a function of these three unknowns then we can solve for the risk-neutral probabilities and fully define Table 3 above.

If we define A_a to be the payoff on Asset A at time one given state ω_a, A_b to be the payoff on Asset A at time one given state ω_b and A_c to be the payoff on Asset A at time one given state ω_c, then the first of our three simultaneous equations using Equation (1) as our guide is...

$$\begin{aligned} A_0 &= \mathbb{E}^Q\left[A_T \times \left(1 + K_b\right)^{-1}\right] \\ A_0 &= \frac{B_0}{B_T}\left(A_a\,q_a + A_b\,q_b + A_c\,q_c\right) \\ \frac{B_T}{B_0}\,A_0 &= A_a\,q_a + A_b\,q_b + A_c\,q_c \end{aligned} \tag{2}$$

If we define D_a to be the payoff on Asset D at time one given state ω_a, D_b to be the payoff on Asset D at time one given state ω_b and D_c to be the payoff on Asset D at time one given state ω_c, then the second of our three simultaneous equations using Equation (1) as our guide is...

$$\begin{aligned} D_0 &= \mathbb{E}^Q\left[D_T \times \left(1 + K_b\right)^{-1}\right] \\ D_0 &= \frac{B_0}{B_T}\left(D_a\,q_a + D_b\,q_b + D_c\,q_c\right) \\ \frac{B_T}{B_0}\,D_0 &= D_a\,q_a + D_b\,q_b + D_c\,q_c \end{aligned} \tag{3}$$

Since probabilities must sum to one then the last of our three simultaneous equations is...

$$q_a + q_b + q_c = 1 \tag{4}$$

Using Table 1 above we will define **matrix A** to be...

$$\mathbf{A} = \begin{bmatrix} A_a & A_b & A_c \\ D_a & D_b & D_c \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 40 & 120 & 150 \\ 40 & 90 & 90 \\ 1 & 1 & 1 \end{bmatrix} \text{ ...where... } \mathbf{A}^{-1} = \begin{bmatrix} 0.0000 & (0.0200) & 1.8000 \\ (0.0333) & 0.0733 & (1.6000) \\ 0.0333 & (0.0533) & 0.8000 \end{bmatrix} \tag{5}$$

We will define **vector v** to be a vector of risk-neutral probabilities. Note that this is the vector that we will solve for. Vector v in vector notation is...

$$\vec{\mathbf{v}} = \begin{bmatrix} q_a \\ q_b \\ q_c \end{bmatrix} \tag{6}$$

Using Table 1 above we will define vector u to be...

$$\vec{\mathbf{u}} = \begin{bmatrix} \frac{B_T}{B_0}\,A_0 \\ \frac{B_T}{B_0}\,D_0 \\ 1 \end{bmatrix} = \begin{bmatrix} 117.60 \\ 85.05 \\ 1 \end{bmatrix} \tag{7}$$

Using Equations (5), (6) and (7) we can write our system of linear equations as a matrix:vector product. The system of linear equations that we must solve is...

$$\mathbf{A}\vec{\mathbf{v}} = \vec{\mathbf{u}} \tag{8}$$

To solve for vector v we multiply both sides of Equation (8) by the inverse of matrix A. Noting that **matrix I** is the identity matrix the solution to vector v is...

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{A}\vec{\mathbf{v}} &= \mathbf{A}^{-1}\vec{\mathbf{u}} \\ \mathbf{I}\vec{\mathbf{v}} &= \mathbf{A}^{-1}\vec{\mathbf{u}} \\ \vec{\mathbf{v}} &= \mathbf{A}^{-1}\vec{\mathbf{u}} \end{aligned} \tag{9}$$

Using Equations (5), (6), (7) and (9) above the solution to vector v is...

$$\vec{\mathbf{v}} = \mathbf{A}^{-1}\vec{\mathbf{u}} = \begin{bmatrix} 0.0000 & (0.0200) & 1.8000 \\ (0.0333) & 0.0733 & (1.6000) \\ 0.0333 & (0.0533) & 0.8000 \end{bmatrix} \begin{bmatrix} 117.60 \\ 85.05 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.0990 \\ 0.7170 \\ 0.1840 \end{bmatrix} \tag{10}$$

The table below presents the **revised** time zero **risk-neutral** probabilities of finding ourselves in either state ω_a, state ω_b or state ω_c at time one...

Table 4 - Revised Risk-Neutral Probabilities (Measure Q)

Description	Symbol	Probability
Risk-neutral probability that we will find ourselves in state ω_a at time one	q_a	0.0990
Risk-neutral probability that we will find ourselves in state ω_b at time one	q_b	0.7170
Risk-neutral probability that we will find ourselves in state ω_c at time one	q_c	0.1840

The Answer To Our Hypothetical Problem

Using risk-neutral pricing Equation (1) and Tables 1, 2 and 4 above the no-arbitrage price of our credit default swap on the company's debt is...

$$S_0 = \frac{B_0}{B_T}\Big(S_a\,q_a + S_b\,q_b + S_c\,q_c\Big) = \frac{100}{105}\Big((50)(0.0990) + (0)(0.7170) + (0)(0.1840)\Big) = \$4.71 \tag{11}$$